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On the Generating Functions of the Young Lattice

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INTRODUCTION

For partitions $\alpha=(\alpha_1, \alpha_2, ...)$ and $\beta=(\beta_1, \beta_2, ...)$, let $\alpha \leqslant \beta$ denote the inclusion relation of the corresponding Young diagrams (as subsets of $\mathbb{Z}^2$); i.e., $\alpha_i \leqslant \beta_i$ $(i=1, 2, ...)$. We call the poset of partitions equipped with this order relation the Young lattice Y [1, p. 17]. For a given partition λ, we consider the "generating function" $f_\lambda(q)=\sum_{\xi \leqslant \lambda} q^{|\xi|}$, where $|\xi|=\xi_1+\xi_2+\cdots$ is the "rank" of ξ, by which Y is a ranked poset. The first and typical purpose of this article is to give an explicit formula for this function (1.4). For a "rectangular" partition $\lambda=(n^k)=(n, ..., n)$ (k times), $f_\lambda(q)$ is known to be a Gaussian coefficient (1.2(3); cf. Macdonald [6, Sect. 12, Ex. 3, pp. 18, 19]). More generally, for two partitions μ and λ, we define the "generating function of the interval $[\mu, \lambda]$" by $f_{\lambda/\mu}(q)=\sum_{\mu \leqslant \xi \leqslant \lambda} q^{|\xi|-|\mu|}$. An explicit formula for $f_{\lambda/\mu}$ is obtained similarly (1.7).

It is natural to generalize $f_{\lambda/\mu}(q)$ to a generating function counting chains in $[\mu, \lambda]$. Let $S=(\xi^{(1)}, ..., \xi^{(r)})$ be a chain of partitions included in this interval; i.e., $\lambda \geqslant \xi^{(1)} \geqslant \cdots \geqslant \xi^{(r)} \geqslant \mu$ $(r \geqslant 0)$. We define $f^{(r)}_{\lambda/\mu}(q)$ to be the sum of $q^{|\xi^{(1)}|-|\mu|} q^{|\xi^{(2)}|-|\mu|} \cdots q^{|\xi^{(r)}|-|\mu|}$ over all such chains. Our main result is 1.10, which expresses $f^{(r)}_{\lambda/\mu}(q)$ in terms of a determinant of the Gaussian coefficient multiplied by a power of q.

Theorem 1.10 has been partly proved by several authors. MacMahon [7, Sects. 495–497] calculated $f^{(r)}_{\lambda/0}(q)$ when λ has no more than 4 parts. Hodge and Pedoe [5, Chap. XIV, Sect. 9] obtained a formula for $f^{(r)}_{\lambda/0}(1)$ as a postulation formula of Schubert variety. In the 1960s Kreweras [6] gave a formula for $f^{(r)}_{\lambda/\mu}(1)$, while Carlitz [2] gave one for $f^{(r)}_{\lambda/0}(q)$. Carlitz transformed the problem in terms of rectangular arrays and used induction on r. His proof is also valid for our case, with only a slight modification; but we

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are rather interested in proving our formula relying upon some combinatorics on Young diagrams and using induction on the numbers of boxes in the skew diagram λ/μ.

$f_{\lambda/\mu}$ has an interpretation as the Poincaré polynomial of a "skew Schubert variety" (cf. [10]).

For a partition λ, take integers k and n such that $\lambda \leqslant (n^k)$, and fix a complete flag $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n+k} = V$ in a $(n+k)$-dimensional vector space V over a field F. If we take F to be the finite field $\mathbb{F}_q$ of q elements, then $f_\lambda(q)$ is exactly the number of k-dimensional subspaces W of V such that $\dim(W \cap V_i) \geqslant \tilde{\lambda}_i$, $1 \leqslant i \leqslant n+k$, where $\tilde{\lambda}_i := \#\{t \mid \lambda_t + k - t + 1 \leqslant i\}$; for example, if $\lambda = (5\,4\,1\,0)$, $k = 4$, $n = 6$, then $\tilde{\lambda} = (1^2\,2^4\,3^2\,4^2)$.

Consider the Grassmannian variety $G_k(\mathbb{C}^{n+k}) = \{k\text{-dimensional subspaces in } \mathbb{C}^{n+k}\}$ over $\mathbb{C}$ and denote the Schubert cell corresponding to λ by C_λ: i.e., $C_\lambda := \{W \in G_k(\mathbb{C}^{n+k}) \mid \dim(W \cap V_i) = \tilde{\lambda}_i,\ 1 \leqslant i \leqslant n+k\}$. Then it holds that the closure $cl(C_\lambda)$ of C_λ includes C_μ iff $\lambda \geqslant \mu$, and that $\dim_{\mathbb{C}} C_\lambda = |\lambda|$; therefore, $f_\lambda(t^2)$ and $f_{\lambda/\mu}(t^2)$ are the Poincaré polynomials of certain subvarieties of $G_k(\mathbb{C}^{n+k})$ (cf. 3.2, 3.3).

It is also possible to generalize these results to the case of flag varieties (4.6), which was communicated to the author from Nagayoshi Iwahori and Takeshi Tokuyama of the University of Tokyo.

1. Definitions and Main Results

1.1. A partition is a sequence of integers $\lambda = (\lambda_1, \lambda_2, ...)$ such that $\lambda_i \geqslant \lambda_{i+1}$, $\lambda_i \geqslant 0$, and $\lambda_i = 0$ for some i. (As for the notations of partitions, we follow [7, Chap. I, Sect. 1, pp. 1–4].) The cardinality of $\{i \mid \lambda_i > 0\}$ is denoted by $l(\lambda)$. We denote the (Young) diagram of λ by the same symbol λ. For two partitions $\alpha = (\alpha_1, \alpha_2, ...)$ and $\beta = (\beta_1, \beta_2, ...)$, we define $\alpha \leqslant \beta$ to mean $\alpha_i \leqslant \beta_i$ for $i = 1, 2, \ldots$.

Definition. For a given partition λ, define:

$$f_\lambda(q) := \sum_{\varphi \leqslant \lambda} q^{|\varphi|},$$

where $|\varphi| := \sum_i \varphi_i$.

1.2. Proposition.

(1) $f_0(q) = 1$, *where* $0 = (0, 0, ...)$.

(2) $f_\lambda(q) = f_{\lambda'}(q)$, *where* λ' *denotes the conjugate partition of* λ.

(3) *If* $\lambda = (m^r) = (m, ..., m)$ (r *times*), *then* $f_\lambda(q)$ *equals the Gaussian binomial coefficient* $\binom{m+r}{r}_q$.

(4) *For* $\lambda=(\lambda_1,\lambda_2,\ldots)$, $l(\lambda)=p$, *put* $\varphi=(\lambda_1-1,\lambda_2-1,\ldots,\lambda_p-1)$ *and* $\theta=(\lambda_1,\lambda_2,\ldots,\lambda_{p-1})$. *Then*

$$f_\lambda(q)=q^p f_\varphi(q)+f_\theta(q).$$

(5) *For* $\lambda=(\lambda_1,\lambda_2,\ldots)$, $\lambda_1=\lambda_2=\cdots=\lambda_k>\lambda_{k+1}$, *put* $\psi=(\lambda_{k+1},\lambda_{k+2},\ldots)$ *and* $\kappa=(\lambda_1,\ldots,\lambda_{k-1},\lambda_k-1,\lambda_{k+1},\ldots)$. *Then*

$$f_\lambda(q)=q^{k\lambda_k}f_\psi(q)+f_\kappa(q).$$

Proof. (1) and (2) are trivial. (3) is shown by (4) and 1.3(2). To get (4), let $\varphi\leqslant\lambda$, and consider the $(p,1)$th box (i.e., the left-bottom extreme box) of the diagram λ. If φ includes it, then it also includes all the boxes on the first column; if it does not, then it excludes all the boxes on the last row. Now split the summation in 1.1 according to these two cases. Similarly, consider the (k,λ_k)th box (i.e., the first corner box) of λ to get (5).

1.3. We collect here some basic formulae for the Gaussian coefficients for the convenience of later use.

LEMMA–DEFINITION. (1) $\binom{m}{n}_q=\binom{m}{m-n}_q$.

(2) $\binom{m}{n}_q=\binom{m-1}{n}_q+q^{m-n}\binom{m-1}{n-1}_q$;

(3) $\binom{m}{n}_q=q^n\binom{m-1}{n}_q+\binom{m-1}{n-1}_q$.

(4) Define $\binom{m}{n}'_q:=q^{n(n-1)/2}\binom{m}{n}_q$, then $\binom{m}{n}'_q=q^n\binom{m-1}{n}'_q+q^{n-1}\binom{m-1}{n-1}'_q$;

(5) $\binom{m}{n}'_q=\binom{m-1}{n}'_q+q^{m-1}\binom{m-1}{n-1}'_q$.

(6) $\sum_{j=a}^{b}q^{j-k}\binom{j}{k}_q=\binom{b+1}{k+1}_q-\binom{a}{k+1}_q$.

Proof. Only (1) and (2) require slight calculation, from which others are straightforward.

1.4. THEOREM (Generating function). *Let n be an integer not less than $l(\lambda)$. Then*

$$f_\lambda(q)=\left|\binom{\lambda_i+1}{i-j+1}'_q\right|_{i,j=1,\ldots,n}.$$

Theorem 1.4 will be proved in Section 2 together with Theorem 1.7.

1.5. DEFINITION. For two partitions λ and μ, define:

$$f_{\lambda/\mu}(q):=\sum_{\lambda\geqslant\xi\geqslant\mu}q^{|\xi-\mu|},$$

where $|\xi-\mu|:=|\xi|-|\mu|$.

1.6. PROPOSITION. (1) $f_{\lambda/\mu}(q)=0$, *unless* $\lambda \geqslant \mu$.

(2) $f_{\lambda/\lambda}(q)=1$.

(3) $f_{\lambda/\mu}(q)=f_{\lambda'/\mu'}(q)$.

(4) $f_{\lambda/0}(q)=f_{\lambda}(q)$.

(5) *Suppose* $\lambda \geqslant \mu$, $\lambda_l > \lambda_{l+1}=0$, *and* $\mu_m > \mu_{m+1}=0$, *and put* $\varphi=(\lambda_1-1, \lambda_2-1, ..., \lambda_l-1)$, $\nu=(\mu_1-1, \mu_2-1, ..., \mu_m-1)$ *and* $\theta=(\lambda_1, \lambda_2, ..., \lambda_{l-1})$. *Then*

$$f_{\lambda/\mu}(q)=q^{l-m}f_{\varphi/\nu}(q)+f_{\theta/\mu}(q).$$

(6) *Suppose* $\lambda \geqslant \mu$ *and* $\lambda_1=\lambda_2=\cdots=\lambda_k>\lambda_{k+1}$, *and put* $\psi=(\lambda_{k+1}, \lambda_{k+2}, ...)$, $\chi=(\mu_{k+1}, \mu_{k+2}, ...)$ *and* $\kappa=(\lambda_1, ..., \lambda_{k-1}, \lambda_k-1, \lambda_{k+1}, ...)$. *Then*

$$f_{\lambda/\mu}(q)=q^{\sum_{j=1}^{k}(\lambda_k-\mu_j)}f_{\psi/\chi}(q)+f_{\kappa/\mu}(q).$$

Proof. (1)–(4) are trivial. To prove (5) and (6), mark the same boxes we mentioned in the proof of 1.2.(4, 5).

1.7. THEOREM (Generating function for the relative case). *Let* $n \geqslant \operatorname{Max}\{l(\lambda), l(\mu)\}$. *Then*

$$f_{\lambda/\mu}(q)=\left| q^{(i-j)\mu_j}\binom{\lambda_i-\mu_j+1}{i-j+1}'_q \right|_{i,j=1,...,n}. \tag{1}$$

1.8. Theorem 1.7 can be generalized to the case of counting chains of length r in the interval $[\mu, \lambda]$ as follows:

DEFINITION.

$$g^{(r)}_{\lambda/\mu}(x_0, x_1, ..., x_r) := \sum_S x_0^{|\xi^{(0)}-\xi^{(1)}|}\cdots x_r^{|\xi^{(r)}-\xi^{(r+1)}|},$$

where $x_0, x_1, ..., x_r$ are $r+1$ indeterminates, and the summation is taken over all sequences of partitions $S=(\xi^{(1)}, ..., \xi^{(r)})$ satisfying $\lambda=\xi^{(0)} \geqslant \xi^{(1)} \geqslant \cdots \geqslant \xi^{(r)} \geqslant \xi^{(r+1)}=\mu$.

Remark. If $\lambda \geqslant \mu$, a skew diagram λ/μ is the set-theoretic difference $\lambda-\mu$. Then $g^{(r)}_{\lambda/\mu}$ can be alternatively written as $\sum_T x^T$, where the summation is taken over all "r-tableaux" T of shape λ/μ, which here means numbered skew diagrams satisfying:

(a) boxes of λ/μ are filled in with integers $0, 1, ..., r$;

(b) the rows and columns are in (weakly) decreasing order, and x^T: $x_0^{m_0(T)}\cdots x_r^{m_r(T)}$, where $m_i(T)$ denotes the number of times the number i

appears in the entries of T. The correspondence between such tableaux T and the sequences of partitions $S=(\xi^{(1)}, ..., \xi^{(r)})$ is the obvious one, by which $m_i(T)=|\xi^{(i)}-\xi^{(i+1)}|$ $(1 \leqslant i \leqslant r)$.

1.9. We specialize $g^{(r)}_{\lambda/\mu}$ to one variable q as follows:

DEFINITION.

$$\begin{aligned} f^{(r)}_{\lambda/\mu}(q) &:= g^{(r)}_{\lambda/\mu}(1, q, ..., q^r) \\ &= \sum_T q^{|T|}, \end{aligned}$$

where $|T| := m_1(T)+2m_2(T)+ \cdots +rm_r(T)$ is the sum of all entries of the tableaux T.

1.10 THEOREM (Generating function for chains). *Let* $n \geqslant \mathrm{Max}\{l(\lambda), l(\mu)\}$. *Then*

$$f^{(r)}_{\lambda/\mu}(q)=\left| q^{(i-j)\mu_j+(i-j)(i-j-1)/2} \binom{\lambda_i-\mu_j+r}{i-j+r}_q \right|_{i,j=1,...,n}. \tag{2}$$

Remark. We note that for the case $l(\lambda)=1$, say $\lambda=(m)$, and $\mu=0$ the right-hand side of (2) coincides with $f_\lambda(q)$ in 1.2(3), as is supposed to by a "transposition" in 3-dimensional Young diagrams.

2. PROOFS OF 1.4 AND 1.7

By 1.6(4), it suffices to prove only 1.7. Let $D(\lambda, \mu; q)$ denote the right-hand side of (1). We claim that it satisfies the same recursive formula as 1.6(6). Let k be the integer as in 1.6(6). Then applying 1.3(5) on the kth row of $D(\lambda, \mu; q)$ we have

$$D(\lambda, \mu; q)=D(\kappa, \mu; q)+|b_{ij}|_{1 \leqslant i,j \leqslant n}$$

in which

$$b_{ij}=q^{(i-j)\mu_j}\binom{\lambda_i-\mu_j+1}{i-j+1}'_q \qquad (i \neq k);$$

$$b_{kj}=q^{\lambda_k-\mu_j+(k-j)\mu_j}\binom{\lambda_k-\mu_j}{k-j}'_q.$$

Since $b_{ij}=0$ for $1 \leqslant i \leqslant k$ and $k+1 \leqslant j \leqslant n$, $\det(b_{ij})_{1 \leqslant i,j \leqslant n}=\det(b_{ij})_{1 \leqslant i,j \leqslant k}\cdot$

$D(\psi, \chi; q)$. Subtracting $q^{-\lambda_k}$ times the $(i+1)$th row of the matrix $(b_{ij})_{1 \leqslant i, j \leqslant k}$ from the ith row (using 1.3(5) again), repeatedly for $i = k-1, k-2, ..., 1$, we get a lower triangular $k \times k$ matrix whose (i, j)-element is

$$q^{\lambda_i - \mu_j + (i-j)\mu_j} \binom{\lambda_i - \mu_j}{i-j}'_q \qquad (1 \leqslant i, j \leqslant k).$$

Hence $|b_{ij}|_{1 \leqslant i, j \leqslant k} = q^{\sum_{j=1}^{k} (\lambda_k - \mu_j)}$.

3. Some Proofs of 1.10

3.1. Our first proof follows that of Carlitz. Carlitz [2] proved 1.10 in case $\mu = 0$ (null partition) by using a recursive formula (in our notation)

$$\begin{aligned} &f_\lambda^{(0)}(q) = 1; \\ &f_\lambda^{(r)}(q) = \sum_{\lambda \geqslant \varphi} q^{|\varphi|} f_\varphi^{(r-1)}(q) \qquad (r \geqslant 1). \end{aligned} \tag{3}$$

Note that if T is a tableau on λ with entries from $\{0, 1, ..., r\}$ then boxes of λ with nonzero entries form a subdiagram, say φ, of λ. This makes (3) clear. Putting $\lambda = (\lambda_1, ..., \lambda_n)$ and $\varphi = (\varphi_1, ..., \varphi_n)$ Carlitz rewrites (3) as

$$\begin{aligned} f_\lambda^{(r)}(q) &= \sum_{\varphi_n = 0}^{\lambda_n} \sum_{\varphi_{n-1} = \varphi_n}^{\lambda_{n-1}} \cdots \sum_{\varphi_1 = \varphi_2}^{\lambda_1} q^{|\varphi|} f_\varphi^{(r-1)}(q) \\ &= \sum_{\varphi_n = 0}^{\lambda_n} q^{\varphi_n} \sum_{\varphi_{n-1} = \varphi_n}^{\lambda_{n-1}} q^{\varphi_{n-1}} \cdots \sum_{\varphi_1 = \varphi_2}^{\lambda_1} q^{\varphi_1} f_\varphi^{(r-1)}(q), \end{aligned}$$

and claims that the right-hand side of (2), when $\mu = 0$, also satisfies this formula.

3.2. In our case, where λ should be replaced by a skew diagram λ/μ, the starting recursive formula must read:

Lemma.

$$\begin{aligned} &f_{\lambda/\mu}^{(0)}(q) = \begin{cases} 1 & (\lambda \geqslant \mu); \\ 0 & (\textit{otherwise}), \end{cases} \\ &f_{\lambda/\mu}^{(r)}(q) = \sum_{\lambda \geqslant \varphi \geqslant \mu} q^{|\varphi - \mu|} f_{\varphi/\mu}^{(r-1)}(q) \qquad (r \geqslant 1). \end{aligned} \tag{4}$$

Noting that $f^{(r-1)}_{\varphi/\mu}(q)=0$ unless $\varphi \geqslant \mu$, (4) can again be rewritten as

$$f^{(r)}_{\lambda/\mu}(q)=\sum_{\lambda \geqslant \varphi} q^{|\varphi-\mu|} f^{(r-1)}_{\varphi/\mu}(q),$$

or putting $\lambda=(\lambda_1, ..., \lambda_n)$ and $\varphi=(\varphi_1, ..., \varphi_n)$,

$$\begin{aligned} f^{(r)}_{\lambda/\mu}(q) &= \sum_{\varphi_n=0}^{\lambda_n} \sum_{\varphi_{n-1}=\varphi_n}^{\lambda_{n-1}} \cdots \sum_{\varphi_1=\varphi_2}^{\lambda_1} q^{|\varphi-\mu|} f^{(r-1)}_{\varphi/\mu}(q) \\ &= q^{-|\mu|} \sum_{\varphi_n=0}^{\lambda_n} q^{\varphi_n} \sum_{\varphi_{n-1}=\varphi_n}^{\lambda_{n-1}} q^{\varphi_{n-1}} \cdots \sum_{\varphi_1=\varphi_2}^{\lambda_1} q^{\varphi_1} f^{(r-1)}_{\varphi/\mu}(q). \end{aligned} \tag{4'}$$

3.3. *Proof of* 1.10 *using* 3.2. It suffices to prove that the right-hand determinant of (2) satisfies the same recursive formula as (4′) above. By induction on r, we may assume that $f^{(r-1)}_{\varphi/\mu}(q)$ is equal to the determinant of the $n \times n$ matrix with its (i, j)-element

$$q^{(i-j)\mu_j+(i-j)(i-j-1)/2} \binom{\varphi_i-\mu_j+r-1}{i-j+r-1}_q.$$

Apply $\sum_{\varphi_i=\varphi_{i+1}}^{\lambda_i} q^{\varphi_i}$ on the ith row of this matrix using 1.3(6) and add the $(i+1)$th row, repeatedly for $i=1, ..., n-1$, and then apply $\sum_{\varphi_n=0}^{\lambda_n} q^{\varphi_n}$ on the nth row. Then the resulting matrix has its (i, j)-element

$$q^{(i-j+1)\mu_j+(i-j)(i-j+1)/2} \binom{\lambda_i-\mu_j+r}{i-j+r}_q$$

whose determinant coincides with $q^{|\mu|}$ times the right-hand side of (2).

3.4. Our second proof relies upon the concept of corner of a diagram.

Definition. Let $\lambda=(\lambda_1, \lambda_2, ...)$ be a partition. A box of the diagram λ is called a corner of λ if the remainder is also a diagram. Obviously, the (i, j)th box is a corner of λ iff $j=\lambda_i$ and $\lambda_i > \lambda_{i+1}$.

3.5. Lemma. *For any two partitions λ and μ,*

$$f^{(r)}_{\lambda/\mu}(q)=q^{|\lambda-\mu|} f^{(r-1)}_{\lambda/\mu}(q)+\sum_{\Omega \supset A \neq \varnothing} (-1)^{|A|+1} f^{(r)}_{\lambda-A/\mu}(q),$$

where Ω denotes the set of all corners of λ and $\lambda-A$ is the diagram obtained from λ by removing all corners which belong to A.

Proof. For each subset A of Ω, let (A) denote the set of all tableaux T on λ/μ with entries from $\{0, 1, ..., r\}$ such that

$$\{(i, j) \in \Omega \mid T(i, j) = 0\} = A,$$

where $T(i, j)$ denotes the (i, j)th entry of T. Then obviously

$$f^{(r)}_{\lambda - B/\mu}(q) = \sum_{A \supset B} \sum_{T \in (A)} q^{|T|},$$

from which we deduce

$$\sum_{T \in (B)} q^{|T|} = \sum_{A \supset B} (-1)^{|A - B|} f^{(r)}_{\lambda - A/\mu}(q)$$

by Möbius inversion. Putting $B = \varnothing$ we have

$$\sum_{T \in (\varnothing)} q^{|T|} = \sum_{A} (-1)^{|A|} f^{(r)}_{\lambda - A/\mu}(q).$$

On the other hand, noting that $T \in (\varnothing)$ iff T has no zero entries, we have

$$\sum_{T \in (\varnothing)} q^{|T|} = q^{|\lambda - \mu|} f^{(r-1)}_{\lambda/\mu}(q),$$

which proves the lemma.

3.6. *Proof of* 1.10 *using* 3.5. Let $D(r; \lambda, \mu; q)$ denote the right-hand side of (2). To prove 1.10 by using 3.5 and induction on r and $|\lambda - \mu|$, it suffices to show that this function satisfies the same formula as 3.5; i.e.,

$$q^{|\lambda - \mu|} D(r-1; \lambda, \mu; q) = \sum_{\Omega \supset A} (-1)^{|A|} D(r; \lambda - A, \mu; q). \tag{5}$$

Let us reconsider a $n \times n$ matrix with its (i, j)th element

$$q^{(i-j)\mu_j + (i-j)(i-j-1)/2}\, q^{\lambda_i - \mu_j - i + j} \binom{\lambda_i - \mu_j + r - 1}{i - j + r - 1}_q. \tag{6}$$

By factoring out $q^{\lambda_i - i}$ from each row and $q^{-\mu_j + j}$ from each column of this matrix, we see that its determinant is equal to the left-hand side of (5). On the other hand, 1.3(2) shows that (6) is equal to

$$q^{(i-j)\mu_j + (i-j)(i-j-1)/2} \binom{\lambda_i - \mu_j + r}{i - j + r}_q$$
$$- q^{(i-j)\mu_j + (i-j)(i-j-1)/2} \binom{(\lambda_i - 1) - \mu_j + r}{i - j + r}_q. \tag{7}$$

Now suppose $\lambda_i = \lambda_{i+1}$. Then it is easily checked that each element on the $(i+1)$th row of (6) is $q^{\lambda_i - 1}$ times the second term of (7). This shows that if the ith row of the diagram λ contains a corner box we can replace the ith row of the determinant of (6) with (7), otherwise with the first term of (7). Then the resulting determinant expands into the right-hand side of (5) by the multi-linearity in its rows.

4. Schubert Varieties—An Interpretation of $f_{\lambda;\mu}(q)$

We show how to read off the meanings of $f_{\lambda;\mu}(q)$ in terms of vector spaces over a field F (cf. [10]).

4.1. Definition. For integers $n, k \geqslant 0$, we let $G_k(F^{n+k})$ be the space of k-dimensional linear subspaces of F^{n+k}. $G_k(F^{n+k})$ is called a Grassmannian variety; it can be identified with the homogeneous space $GL(n+k)/P(k,n)$, where $P(k,n)$ is the parabolic subgroup of $GL(k+n)$ corresponding to (k,n). Let $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n+k} = F^{n+k}$ be a fixed complete flag, and define for each subspace W belonging to $G_k(F^{n+k})$ a partition $\lambda(W) = (\lambda_1(W), ..., \lambda_k(W))$ by

$$\lambda_{k-i+1}(W) := \operatorname{Min}\{j \mid \dim(W \cap V_j) = i\} - i$$

for $1 \leqslant i \leqslant k$. Then $\lambda \leqslant (n^k)$. Conversely, for each $\lambda \leqslant (n^k)$, we let $C_\lambda := \{W \in G_k(F^{n+k}) \mid \lambda(W) = \lambda\}$. Each C_λ is called a Schubert cell.

4.2. Proposition (cf. [2, Chap. III, Sect. 3]). *Let $F = \mathbb{C}$, then*

(1) *C_λ is homeomorphic to $\mathbb{C}^{|\lambda|}$;*

(2) *$\mathrm{cl}(C_\lambda) \supset C_\mu$ iff $\lambda \geqslant \mu$, in which $\mathrm{cl}(C_\lambda)$ denotes the closure of C_λ;*

(3) *$\mathrm{cl}(C_\lambda) = \bigsqcup_{\varphi \leqslant \lambda} C_\varphi$.*

4.3. Corollary. *$f_\lambda(t^2)$ is the Poincaré polynomial of $\mathrm{cl}(C_\lambda)$. $f_{\lambda;\mu}(t^2)$ is the Poincaré polynomial of $\bigsqcup_{\mu \geqslant \varphi \geqslant \lambda} C_\varphi$.*

Remark. $\mathrm{cl}(C_\lambda)$ or $\bigsqcup_{\mu \leqslant \varphi \leqslant \lambda} C_\varphi$ is the space of k-spaces W in F^{n+k} satisfying $\dim(W \cap V_i) \geqslant \tilde{\lambda}_i$ or $\tilde{\mu}_i \geqslant \dim(W \cap V_i) \geqslant \tilde{\lambda}_i$ $(1 \leqslant i \leqslant n+k)$, respectively, where $\tilde{\lambda}$ and $\tilde{\mu}$ are defined as in the Introduction.

4.4. Let M be the space of $k \times (n+k)$ F-matrices of rank k, and M_λ consist of the elements $A = (a_{ij})$ of M satisfying $a_{ij} = 0$ for all i, j such that $\lambda_i + k - i + 1 > j$. $GL(k)$ acts on M by the matrix multiplication from the left and there is a $1-1$ correspondence $\Omega \mapsto W_\Omega$ between the orbits of this action and the elements of $G_k(F^{n+k})$, where W_Ω is the subspace of F^{n+k}

spanned by the rows of a matrix in the orbit Ω. Let Ω_λ be the orbit such that W_{Ω_λ} is in C_λ; then the closure $\mathrm{cl}(C_\lambda)$ of C_λ includes C_μ iff Ω_μ intersects with M_λ. Thus

PROPOSITION. $\mathrm{cl}(C_\lambda)$ *can be identified with* $GL(k)\backslash GL(k)\cdot M_\lambda$.

4.5. PROPOSITION. *Let F be the finite field of q elements. Then* $f_\lambda(q)$ *and* $f_{\lambda/\mu}(q)$ *are the numbers of points in* $\mathrm{cl}(C_\lambda)$ *and* $\bigsqcup_{\mu\leqslant\varphi\leqslant\lambda} C_\varphi$, *respectively.*

4.6. *Remark.* Similar argument is also valid for flag varieties. Let P be the parabolic subgroup of $GL(N)$ corresponding to $(k_1, k_2, ..., k_r)$ where $k_1+k_2+\cdots k_r=N$. If $F=\mathbb{F}_q$, the number of points in $GL(N)/P$; i.e., $\binom{k_1+k_2+\cdots+k_r}{k_1,k_2,...,k_r}_q$ is equal to the generating polynomial of the lattice of $(r-1)$-tuples of partitions $\lambda=(\lambda^{(2)}, ..., \lambda^{(r)})$ such that the diagram of each $\lambda^{(i)}$ is contained in the $k_i\times(k_1+\cdots+k_{i-1})$-rectangle, with rank $|\lambda|=|\lambda^{(2)}|+\cdots+|\lambda^{(r)}|$.

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